NORTH-HOLLAND

Toughness and Spectrum of a Graph

A. E. Brouwer
Department of Mathematics
Technical University of Eindhoven
Eindhoven, The Netherlands

Dedicated to J. J. Seidel on the occasion of his 75th birthday

Submitted by Aart Blokhuis

ABSTRACT

We derive (asymptotically best possible) lower bounds for the toughness of a graph in terms of its eigenvalues.

The *toughness* of a finite connected graph Γ with vertex set X is defined as the minimum of the quotient $|Z|/c(X \setminus Z)$ over all subsets Z of X such that $c(X \setminus Z) > 1$, where $c(A)$ denotes the number of connected components of the graph induced on A by Γ. For example, the toughness of the Petersen graph equals $4/3$, and the unique way to achieve this quotient is by taking for Z a 4-coclique (and then $X \setminus Z$ carries the structure of $3K_2$).

Alon [1] showed how to derive lower bounds for the toughness t of a graph Γ given its spectrum, and independently, but later, I found similar results. However, by combining the two approaches it is possible to prove the best possible result (up to an additive constant).

THEOREM 0.1. *Let Γ be a connected noncomplete regular graph of valency d and let λ be the maximum of the absolute values of the eigenvalues of Γ distinct from d. Then the toughness t of Γ satisfies $t > d/\lambda - 2$.*

The constant 2 can be improved a little, but not much, since there are infinitely many examples of graphs wit $t \leqslant d/\lambda$. Indeed, for strongly regular graphs with smallest eigenvalue s the Delsarte upper bound for the size of a

655 Avenue of the Americas, New York, NY 10010
0024-3795/95/$9.50
SSDI 0024-3795(95)00154-J

coclique is $|C| \leqslant |X|/(1 + d/(-s))$. When equality holds, we can take $Z = X \setminus C$ and find $t \leqslant d/(-s)$. Now there are many cases where this upper bound for the size of a coclique is met with equality; for example, if we take for Γ the complement of the collinearity graph of a generalized quadrangle $GQ(q, q)$ (q a prime power), then the three distinct eigenvalues are $d = q^3$, $r = q$, and $s = -q$ so that $\lambda = q$, and we find $q^2 - 2 < t \leqslant q^2$. The example of the Petersen graph above shows that t can be actually less than d/λ (it has $d = 3$ and $\lambda = 2$).

1. INTERLACING LEMMAS

We use interlacing in two forms: (i) The eigenvalues of a subgraph (all our subgraphs are induced) of Γ interlace those of Γ and (ii) given a partition Π of the index set of a matrix A, define a matrix B by: $B_{R,S}$ equals the average row sum of the block of A with rows indexed by R and columns by S (where $R, S \in \Pi$). Then the eigenvalues of B interlace those of A. (These results are well known; see, e.g. [2]–[5]. More can be said in the case of equality.)

Applying (ii) to the adjacency matrix A of Γ, with the trivial partition having one part only, we find the following very well known lemma.

LEMMA 1.1. *The average valency of a graph is not more than its largest eigenvalue.*

[Applying (ii) to A^2 we get the slightly better lemma: The average squared valency of a graph is not more than the square of its largest eigenvalue.]

Now assume that Γ has eigenvalues $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$. Then by (i) the eigenvalues of the subgraph Δ induced on a subset Y interlace those of Γ. In particular, we find from the above lemma that at most one component of Δ has average valency larger than θ_2.

Next, let us apply (ii) to a matrix M of order n with constant row and column sums k, with a partition $\Pi = \{R, S\}$ into two parts of sizes $r, n - r$, and let the (R, R) block of M have the average row sum a. The 2×2 matrix B now has eigenvalues k and $(na - rk)/(n - r)$, so that the latter value lies between the smallest and the second largest eigenvalue of M. In particular, we find, taking $M = A$ and $M = A^2$, the following lemmas.

LEMMA 1.2. *Let Γ be regular of degree d on n vertices and let the graph induced on the r-set R have average valency e. Then $\theta_2 \geqslant (ne - rd)/(n - r) \geqslant \theta_n$ and hence $r \leqslant n(e - \theta_n)/(d - \theta_n)$.*

LEMMA 1.3. *Let* Γ *be regular of degree* d *on* n *vertices and put* $\lambda = \max(|\theta_2|, |\theta_n|)$. *Then* $\sum_x(|\Gamma(x) \cap R|)^2 - d^2r^2/n \leqslant \lambda^2 r(n-r)/n$.

[Indeed, the sum of all entries of the matrix A^2 in the (R, R) block equals the number of paths $\gamma \sim \xi \sim \delta$, with $\gamma, \delta \in R$ and $\xi \in X$, that is, $\sum_x(|\Gamma(x) \cap R|)^2$.]

This last inequality can also be written as

$$\sum_x \left(|\Gamma(x) \cap R| - \frac{rd}{n}\right)^2 \leqslant \frac{r(n-r)}{n}\lambda^2.$$

2. PROOF OF THE THEOREM

Let the graph Γ have toughness t and let Z be a subset of the vertex set X such that $|Z| = tc$, where $c > 1$ is the number of connected components of the graph induced on $Y := X \setminus Z$. By Lemma 1.2 (applied to a coclique obtained by choosing one point from each component of Y), we have $c \leqslant \lambda n/(d + \lambda)$. As we saw in the previous section, at most one component of Y, B say, has average valency more than λ; put $B := \varnothing$ if there is no such component. Put $C := Y \setminus B$. If $|Y| \leqslant \lambda n/d$, then $|Z| \geqslant (d - \lambda)n/d$ so that $t \geqslant (d^2 - \lambda^2)/(d\lambda) \geqslant d/\lambda - 1$. So we may assume that $|Y| > \lambda n/d$. By Lemma 1.3 applied to Y we find

$$|C|.\left(\frac{d|Y|}{n} - \lambda\right)^2 \leqslant \sum_{x \in C}\left(|\Gamma(x) \cap Y| - \frac{d|Y|}{n}\right)^2 \leqslant \lambda^2|Y|\left(1 - \frac{|Y|}{n}\right). \quad (1)$$

Put $\tau := |Y|/n = 1 - tc/n$. Then (1) becomes

$$|C|.\left(\tau - \frac{\lambda}{d}\right)^2 \leqslant \frac{\lambda^2}{d^2}\tau tc,$$

where $\lambda/d < \tau < 1$. Thus,

$$t \geqslant \frac{|C|}{\tau c}\left(\frac{d}{\lambda}\tau - 1\right)^2. \quad (2)$$

We may assume that $\lambda/d < 1/2$; otherwise, there is nothing to prove.

Assume $|C| \geqslant c$ and $t \leqslant d/\lambda - 2$. Then, using $c \leqslant \lambda n/(d-\lambda)$, we have $\tau = 1 - ct/n \geqslant 3\lambda/(d+\lambda) > \lambda/d$. Also, by (2), $t \geqslant (d\tau/\lambda - 1)^2/\tau$. Since the function $f(x) = d^2x/\lambda^2 + 1/x$ is increasing for $x \geqslant \lambda/d$, we find $d/\lambda - 2 \geqslant t \geqslant d^2\tau/\lambda^2 + 1/\tau - 2d/\lambda \geqslant 3d^2/(\lambda(d+\lambda)) + (d+\lambda)/3\lambda - 2d/\lambda = (4d^2 - 4\,d\lambda + \lambda^2)/(3\lambda(d+\lambda)) > (3d^2 - 3\,d\lambda - 6\lambda^2)/(3\lambda(d+\lambda)) = \mathrm{d}/\lambda - 2$, *a contradiction.*

That leaves us with the case $|C| = c - 1$, B is nonempty, and all components of C are singletons. Now we have, instead of (1), the sharper

$$|C|.\left(\frac{d|Y|}{n}\right)^2 \leqslant \lambda^2|Y|\left(1 - \frac{|Y|}{n}\right), \tag{3}$$

i.e.,

$$t \geqslant \tau\left(1 - \frac{1}{c}\right)\frac{d^2}{\lambda^2} \geqslant \tau\frac{d^2}{2\lambda^2}.$$

If $\tau < 2\lambda/(d+\lambda)$, the $tc/n > (d-\lambda)/(d+\lambda)$, and since $c/n \leqslant \lambda/(d+\lambda)$, it follows that $t > d/\lambda - 1$. Thus, we may assume that $\tau \geqslant 2\lambda/(d+\lambda)$. However, then $t \geqslant d^2/\lambda(d+\lambda) > (d-\lambda)/\lambda = d/\lambda - 1$. ■

3. REMARKS

The above estimates may be improved by noticing that either $|C|/c$ is much larger than 1 or the average valency of C is much less than λ (but the computations become messy, and I have not pursued this).

[For example, assume that C has average valency $\gamma\lambda$, where $0 \leqslant \gamma \leqslant 1/2$. Instead of (2), we now find

$$t \geqslant \frac{|C|}{\tau c}\left(\frac{d}{\lambda}\tau - \gamma\right)^2.$$

If we want to prove $t > d/\lambda - 1$, then we may assume as above that $\tau \geqslant 2\lambda/(d+\lambda)$ and $|C| \geqslant c$, so

$$t \geqslant \frac{d+\lambda}{2\lambda}\left(\frac{2d}{d+\lambda} - \frac{1}{2}\right)^2,$$

i.e.,

$$t - \left(\frac{d}{\lambda} - 1\right) \geqslant \frac{2\lambda}{d + \lambda} + \frac{d + \lambda}{8\lambda} - 1 \geqslant 0.$$

However one can check that it is impossible that equality holds in all estimates, so if C has average valency at most $\lambda/2$, then indeed $t > d/\lambda - 1$.]

On the other hand, since $d \geqslant \lambda$ and t can be arbitrarily close to zero, in a result of the form $t \geqslant d/\lambda - a$ for some constant a, we must have $a \geqslant 1$ if the result holds for arbitrary regular graphs. (By the way, regularity is not very essential in the theorem—similar results hold for arbitrary graphs.) Stronger results can also be obtained by keeping θ_2 and θ_n separate, instead of lumping them together in λ.

This note would never have been written had I not read the last chapter of Jan van den Heuvel's thesis [6], which describes the contents of a paper by Bauer, van den Heuvel, and Schmeichel, where triangle-free t-tough graphs are constructed for arbitrarily large t, thus destroying a conjecture by Chvátal. I am also indebted to Jan van den Heuvel for providing me with a copy of [1]. Willem Haemers taught me interlacing and made several useful remarks regarding the presentation here. Also the referee suggested some improvements in the presentation.

REFERENCES

1 N. Alon, Tough Ramsey graphs without short cycles, Preprint, 1993.
2 A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer, Heidelberg, 1989.
3 D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs*, VEB, Berlin, 1979; Academic, New York, 1980.
4 W. H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Reidel, Dordrecht, 1980.
5 W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* 226–228:593–616 (1995).
6 J. van den Heuvel, Degree and Toughness Conditions for Cycles in Graphs, Ph.D. Thesis, Twente Univ., 1993.

Received 8 November 1993; final manuscript accepted 9 February 1995